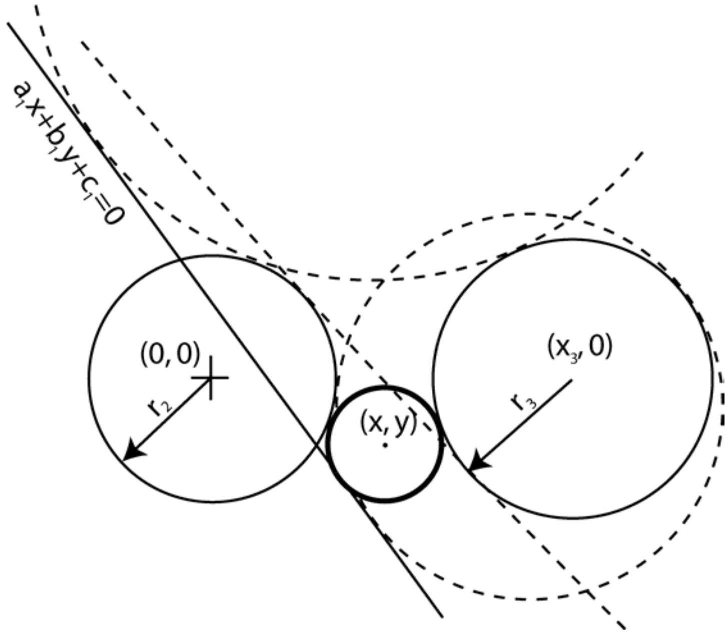
Parameters of a Circle Tangent to a Line and Two Circles

Given a line having the equation $a_1x + b_1y + c_1 = 0$; and two circles translated so that one circle is centred at (0, 0) and has radius r_2, and the other circle is centred at *$(x_3, 0)$* and has radius r_3. It is required to find the circle tangential to these elements, centred at *(x, y)* radius *r*.
There may be up to eight solutions when all elements intersect.



The line parallel to a line $ax + by + c = 0$ at a distance *d* is given by,

$$ax + by + c - d\sqrt{(a^2 + b^{2)}} = 0$$

If the line equation is normalized, this reduces to, $ax + by + c - d = 0$

The necessary equations for tangentiality are then,

$$a_1x + b_1y + c_1 - rt = 0 \qquad \ldots\text{(i)}$$

$$x^2 + y^2 = (r_2 + r)^2 \qquad \ldots\text{(ii)}$$

$$(x_3 - x)^2 + y^2 = (r_3 + r)^2 \qquad \ldots\text{(iii)}$$

where t=±1

rearranging (i), $$r = \frac{a_1x + b_1y + c_1}{t} \qquad \ldots\text{(iv)}$$

subtracting (iii) from (ii),

$$-x_3{}^2 + 2xx_3 = r_2{}^2 - r_3{}^2 + 2r(r_2 - r_3) \qquad \ldots\text{(v)}$$

substituting (iv) in (v) and rearranging,

$$2x(a_1(r_2 - r_3) - x_3t) + 2yb_1(r_2 - r_3) + 2c_1(r_2 - r_3) + t(r_2{}^2 - r_3{}^2 + x_3{}^2) = 0$$

extract the constants,

$$a = 2(a_1(r_2 - r_3) - x_3t)$$

$$b = 2b_1(r_2 - r_3)$$

$$c = 2c_1(r_2 - r_3) + t(r_2{}^2 - r_3{}^2 + x_3{}^2)$$

then, $ax + by + c = 0$ ...(vi)

use (vi) to eliminate y from (i) giving, $x = \dfrac{b_1 c - bc_1 + rtb}{a_1 b - ab_1}$

extract the constants,

$u = b_1 c - bc_1$

$s = a_1 b - ab_1$

then, $x = \dfrac{u + rtb}{s}$ ...(vii)

substituting (vi) in (ii) gives,

$x^2(a^2 + b^2) + 2xac + c^2 - b^2(r_2 + r)^2 = 0$ ...(viii)

substituting (vii) in (viii) and rearranging gives,

$r^2(t^2b^2(a^2 + b^2) - b^2s^2) + 2r(utb(a^2 + b^2) + acstb - b^2s^2r_2) + u^2(a^2 + b^2) + 2acsu + c^2s^2 - b^2s^2r_2^{\ 2} = 0$

extract the constants,

$A = (t^2b^2(a^2 + b^2) - b^2s^2)$

$B = 2(utb(a^2 + b^2) + acstb - b^2s^2r_2)$

$C = u^2(a^2 + b^2) + 2acsu + c^2s^2 - b^2s^2r_2^{\ 2}$

then, $Ar^2 + Br + C = 0$

thus, $r = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ the real and positive solutions being the circle radii

subtract (iii) from (ii) to obtain,

$x = \dfrac{(r_2 + r)^2 - (r_3 + r)^2 + x_3^{\ 2}}{2x_3}$ ($x_3 \neq 0$, i.e. circles not concentric)

substitute in (i) to obtain y ($b_1 \neq 0$)
Negate t, r_2 and r_3 to get other solutions
Note that if b=0 in (vi), eliminate x rather than y.