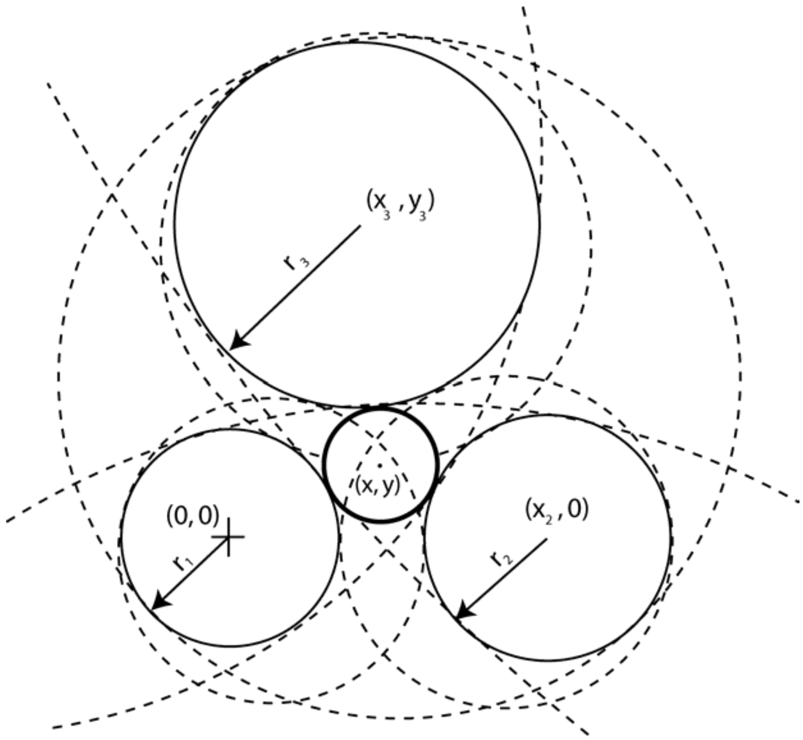
Parameters of a Circle Tangent to Three Other Circles

Given three circles, translated and rotated so that the first circle is centred at *(0, 0)*, radius r_1; the second circle is centred at $(x_2, 0)$, radius r_2; and the third circle is centred at (x_3, y_3), radius r_3. It is required to find the circle tangential to these circles, centred at *(x, y)*, radius *r*.
There may be up to eight solutions.



The necessary equations for tangentiality are:-

$$x^2 + y^2 = (r + r_1)^2 \qquad \text{....(i)}$$

$$(x_2 - x)^2 + y^2 = (r + r_2)^2 \qquad \text{...(ii)}$$

$$(x_3 - x)^2 + (y_3 - y)^2 = (r + r_3)^2 \qquad \text{...(iii)}$$

from (i), $\quad r = \sqrt{x^2 + y^2} - r_1$

substituting for *r* in (ii) and rearranging,

$$2\sqrt{x^2 + y^2} = \frac{-2xx_3 + x_2{}^2 - (r_2 - r_1)^2}{(r_2 - r_1)} \qquad \text{...(iv)}$$

substituting for r in (iii) and rearranging,

$$2\sqrt{x^2 + y^2} = \frac{-2xx_3 - 2yy_3 + x_3{}^2 + y_3{}^2 - (r_3 - r_1)^2}{(r_3 - r_1)} \qquad \text{...(v)}$$

equating (iv) and (v) and rearranging,

$$2x(x_2(r_3 - r_1) - x_3(r_2 - r_1)) + 2yy_3(r_1 - r_2) + (r_2 - r_1)(x_3{}^2 + y_3{}^2 - (r_3 - r_1)^2) - (r_3 - r_1)(x_2{}^2 - (r_2 - r_1)^2) = 0$$

extract the constants,

$$a = 2(x_2(r_3 - r_1) - x_3(r_2 - r_1))$$

$$b = 2y_3(r_1 - r_2)$$

$$c = (r_2 - r_1)(x_3{}^2 + y_3{}^2 - (r_3 - r_1)^2) - (r_3 - r_1)(x_2{}^2 - (r_2 - r_1)^2)$$

then, $ax + by + c = 0 \quad ....(vi)$

squaring (vi) and rearranging (i) and (ii),

$$b^2 y^2 = a^2 x^2 + 2axc + c^2 \quad ...(vii)$$

$$y^2 = -x^2 + (r + r_1)^2 \quad ...(viii)$$

$$y^2 = -x^2 + 2xx_2 - x_2^{\ 2} + (r + r^2)^2 \quad ...(ix)$$

subtracting (viii) from (vii) and (viii) from (ix) gives,

$$x^2(a^2 + b^2) + 2axc + c^2 - b^2(r + r_1)^2 = 0 \quad ...(x)$$

$$2xx_2 - x_2^{\ 2} + (r + r_2)^2 - (r + r_1)^2 = 0 \quad ...(xi)$$

from (xi), $x = \dfrac{2r(r_1 - r_2) + (x_2^{\ 2} + r_1^{\ 2} - r_2^{\ 2})}{2x_2}$

let, $t = \dfrac{(x_2^{\ 2} + r_1^{\ 2} - r_2^{\ 2})}{2}$

then, $x = \dfrac{r(r_1 - r_2) + t}{x_2} \quad ...(xii)$

substituting (xii) in (x),

$$\frac{(r(r_1 - r_2) + t)^2 (a^2 + b^2)}{x_2^{\ 2}} + \frac{2ac(r(r_1 - r_2) + t)}{x_2} + c^2 - b^2 r^2 - 2rr_1 b^2 - b^2 r_1^{\ 2} = 0$$

rearranging,

$$r^2((r_1 - r_2)^2(a^2 + b^2) - x_2^{\ 2} b^2) + r2(t(r_1 - r_2)(a^2 + b^2) + acx_2(r_1 - r_2) - r_1 x_2^{\ 2} b^2) +$$
$$t^2(a^2 + b^2) + 2acx_2 t + c^2 x_2^{\ 2} - r_1^{\ 2} x_2^{\ 2} b^2 = 0$$

extract the constants,

$$A = (r_1 - r_2)^2(a^2 + b^2) - x_2^{\ 2} b^2$$

$$B = 2(t(r_1 - r_2)(a^2 + b^2) + acx_2(r_1 - r_2) - r_1 x_2^{\ 2} b^2)$$

$$C = t^2(a^2 + b^2) + 2acx_2 t + c^2 x_2^{\ 2} - r_1^{\ 2} x_2^{\ 2} b^2$$

then, $Ar^2 + Br + C = 0$

thus, $r = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ the real and positive solutions being the circle radii

substitute in (xii) to find x ($x_2 \neq 0$) substitute in (vi) to find y.
Negate r_n to find alternative solutions.
If the first two given circles are concentric, (xii) is undefined. In this case, exchange the circles.

If $r_1 = r_2$, then $b = 0$. Then $A = (r_1 - r_2)^2 a^2$ and is therefore also zero.

In this case, we can equate (i) and (ii) to find, $x = \dfrac{x_2}{2}$.

Subtracting (iii) from (i), $y = \dfrac{2r(r_1 - r_3) + r_1^{\ 2} - r_3^{\ 2} + x_3^{\ 2} + y_3^{\ 2} - 2xx_3}{2y_3}$

Let $k = r_1^2 - r_3^2 + x_3^2 + y_3^2 - 2xx_3$

Then, $y = \dfrac{2r(r_1 - r_3) + k}{2y_3}$ ...(xiii)

Substitute for y in (i) and rearrange to give,

$4r^2((r_1 - r_3)^2 - y_3^2) + 4r(k(r_1 - r_3) - 2y_3^2 r_1) + 4x^2 y_3^2 + k^2 - 4y_3 r_1^2 = 0$

Extract the constants,

$A = 4((r_1 - r_3)^2 - y_3^2)$

$B = 4(k(r_1 - r_3) - 2y_3^2 r_1)$

$C = 4x^2 y_3^2 + k^2 - 4y_3 r_1^2$

then, $Ar^2 + Br + C = 0$

thus, $r = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ the real, positive solutions being the circle radii

Substitute in (xiii) to find y.