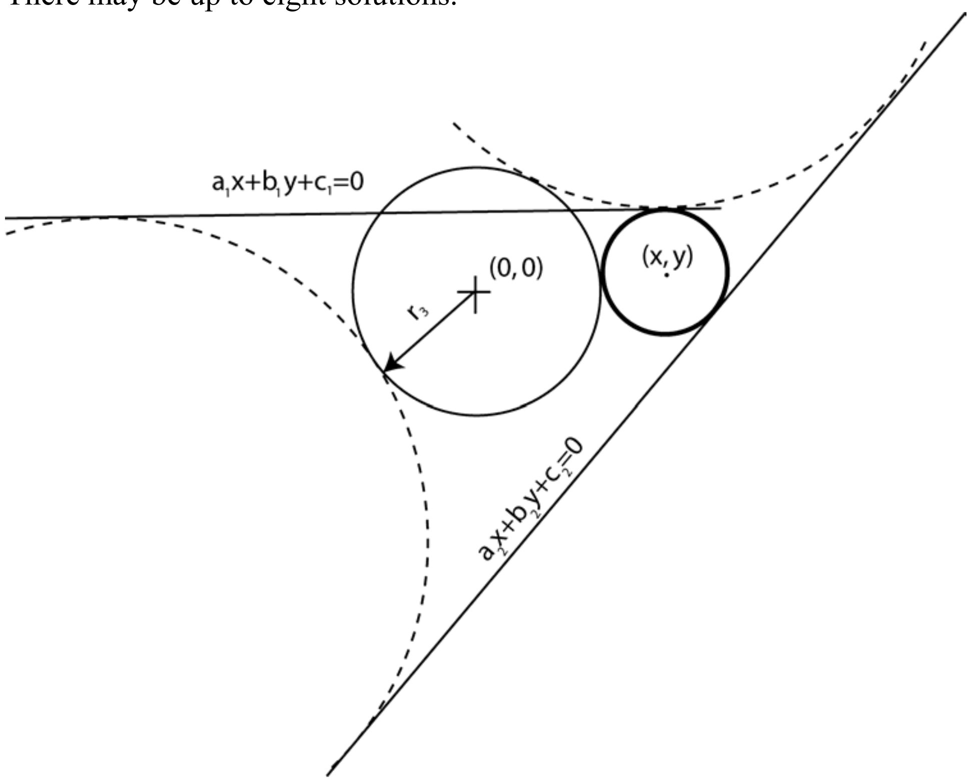
Parameters of a Circle Tangent to Two Lines and a Circle

Given a line having the equation $a_1x + b_1y + c_1 = 0$; a line having the equation $a_2x + b_2y + c_2 = 0$; and a circle centred on (0, 0) with radius r_3, it is required to find the circle tangential to these elements, centred at *(x, y)* radius *r*.
There may be up to eight solutions.



The line parallel to a line $ax + by + c = 0$ at a distance *d* is given by,

$$ax + by + c - d\sqrt{(a^2 + b^{2)}} = 0$$

If the line equation is normalized, this reduces to,

$$ax + by + c - d = 0$$

The necessary equations for tangentiality are then,

$$a_1x = b_1y + c_1 - rt_1 = 0 \quad \ldots\text{(i)}$$

$$a_2x + b_2y + c_2 - rt_2 = 0 \quad \ldots\text{(ii)}$$

$$x^2 + y^2 = (r_3 + r)^2 \quad \ldots\text{(iii)}$$

where $t_n = \pm 1$

from (i) $y = \dfrac{rt_1 - a_1x - c_1}{b_1} \quad \ldots\text{(iv)}$

substituting (iv) in (ii) and rearranging,

$$x = \frac{r(t_1b_2 - t_2b_1) + b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

extract the constants,

$u = t_1 b_2 - t_2 b_1$

$w = b_1 c_2 - b_2 c_1$

$s = a_1 b_2 - a_2 b_1$

then, $x = \dfrac{ru + w}{s}$ ...(v)

substituting (iv) in (iii) and rearranging,

$x^2(a_1^{\ 2} + b_1^{\ 2}) + 2a_1 x(c_1 - rt_1) + (rt_1 - c_1)^2 - b_1^{\ 2}(r_3 + r)^2 = 0$

as line is normalized, $(a_1^{\ 2} + b_1^{\ 2}) = 1$, so

$x^2 + 2a_1 x(c_1 - rt_1) + (rt_1 - c_1)^2 - b_1^{\ 2}(r_3 + r)^2 = 0$ ...(vi)

substituting (v) in (vi) and rearranging,

$$r^2(u^2 - 2a_1 sut_1 + t_1^{\ 2}s^2 - b_1^{\ 2}s^2) + 2r(uw + c_1 a_1 su - a_1 st_1 w - c_1 t_1 s^2 - r_3 b_1^{\ 2} s^2) +$$
$$(w^2 + 2a_1 sc_1 w + c_1^{\ 2}s^2 - b_1^{\ 2} r_3^{\ 2} s^2) = 0 \quad \text{...(vii)}$$

extract the constants,

$A = u^2 - 2a_1 sut_1 + t_1^{\ 2}s^2 - b_1^{\ 2}s^2$

$B = 2(uw + c_1 a_1 su - a_1 st_1 w - c_1 t_1 s^2 - r_3 b_1^{\ 2} s^2)$

$C = w^2 + 2a_1 sc_1 w + c_1^{\ 2}s^2 - b_1^{\ 2} r_3^{\ 2} s^2$

then, $Ar^2 + Br + C = 0$

thus, $r = \dfrac{-B \pm \sqrt{B^2 - 4C}}{2A}$ the real and positive solutions being the circle radii

Substitute back in (v) to find x if $s \neq 0$, that is, lines not parallel, otherwise in (vi)
To find all values, negate t_1, t_2 and r_3.
Note that if the first line is vertical, (iv) is invalid. In this case, exchange the lines.